Optimal Estimation Of The Roughness In Open Channel Flows

A Thesis Submitted
in Partial Fulfillment of the Requirements
for the Degree of

MASTER OF TECHNOLOGY

by

RAMESH. R

to the

DEPARTMENT OF CIVIL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

June, 1997

Dedicated

to

MY BELOVED PARENTS

CERTIFICATE

It is certified that the work contained in the thesis entitled "*Optimal Estimation of the Roughness in open Channel Flows*", by *Ramesh R*, has been carried out under our supervision and that this work has not been submitted elsewhere for a degree.

Dr. Bithin Datta
Professor
Department of Civil Engineering
I.I.T., Kanpur

Dr. B.S. Murty
Associate Professor
Department of Civil Engineering
I.I.T., Kanpur

June, 1997.

ACKNOWLEDGEMENT

I express my sincere gratitude to my thesis supervisors **Dr. Murty B.S** and **Dr. Bithin Datta** for their guidance and support during my thesis and course work. I am also thankful to other Faculty members of the Hydraulics and Water Resources Department for their support in my course work.

I am also thankful to all my friends here who have made my stay here pleasant and memorable. I am thankful to Bhole and co. for they had been my constant companion in C.C and made my work enjoyable.

Abstract

The inverse problem of estimating the open channel flow roughness is solved using an embedded optimization model. Measured data for flow depths and discharges at different locations and times are used as inputs to the optimization model. The nonlinear optimization model embeds the finite-difference approximations of the governing equations for unsteady flow in Open-Channel as equality constraints. The Sequential Quadratic Programming Algorithm is used to solve the Optimization model. The performance of the proposed parameter estimation model is evaluated for different scenarios of data availability and noise in flow measurement data. Solution results for illustrative problems indicate the potential applicability of the proposed model.

Contents

List of Figures

List of Tables

Nomenclature

n	=	Manning's coefficient.
h	=	Depth of flow in the channel.
q	=	Dischage per unit width of the channel.
U	=	Velocity of flow in the channel.
s_o	=	Bottom slope of the channel.
s_f	=	Friction slope.
k	=	Conveyance of the channel.
g	=	Accleration due to gravity.
t	=	Time.
x	=	Longitudinal distance in metres.
Δx	=	Grid spacing
Δt	=	Computational time step.
f	=	Any variable s_f, q or h
j	=	Grid point in space.
k	=	Time level.
θ	=	Weighting coefficient in Priessmann scheme.
α	=	Induced error percent.
N	=	Total number of nodes.
p	=	Total number of time steps.
$x(o)$	=	Simulated value of depth or discharge from optimizer.
$x(m)$	=	Observed value of depth or discharge.
x_i	=	Simulated noise free measurement data
x_i'	=	Erroneous measurement data
ϵ_i	=	A random error term sampled from a uniform distribution

Chapter 1

Introduction

1.1 Overview and Empirical Methods

An accurate estimation of Manning's roughness coefficient is of primary importance in any study involving open channel flows such as:

- Flood estimation,routing and damage mitigation,
- Optimal design and operation of canal and irrigation systems,
- Estimation of aggradation and degradation of rivers due to natural and man made causes,
- Utilization of surface water resources and other river engineering studies.

Estimation of Manning's coefficient for a natural channel is not a trivial task as it depends on several factors like surface roughness, vegetation,channel irregularity,obstructions,channel alignment, sedimentation and scouring. Many empirical procedures (French,1985) have been suggested in the past for estimating the value of "n",the Manning's coefficient. The soil conservation service method (SCS)(Urquhart,1975) for estimating "n" involves selection of a basic

value of "n" corresponding to uniform,straight and regular channel in native material and then modifying its value by adding correction factors. The correction factors are determined by a heuristic method by consideration of some of the above mentioned factors.

A second method for estimating "n" utilizes of standard tables. Chow (1959) presented an extensive table containing a minimum, normal, and a maximum value of "n" for various types of channels. This method gives a satisfactory estimate for man made channels, but is not suitable for natural channels. According to U.S Geological Survey, the photographs of channels of known resistance along with the knowledge of hydraulic and geometric parameters which define the channel for a specified flow rate can be useful in estimating the resistance coefficient (Barnes,1967). The U.S Geological survey also maintains a program which trains the engineers in the estimation of "n". However, this is not a formal procedure and the accuracy of estimation depends on the past experience of the user.

From a theoretical viewpoint, value of the resistance coefficient can be estimated from velocity and depth measurements which depend upon roughness. Chow(1959), suggested that Manning's "n"can be estimated with the help of velocity and depth measurements at a cross-section using an empirical formula given as follows.

$$n = \frac{(x-1)h^{1/6}}{6.78(x+0.95)} \tag{1.1}$$

where

$$x = U_{0.2}/U_{0.8} \tag{1.2}$$

$U_{0.2}$ = velocity at $2/10^{th}$ of depth
$U_{0.8}$ = velocity at $8/10^{th}$ of depth.
h = depth of flow.
The above equations (1.1) and(1.2) are applicable only when the channel is

wide and velocity measurements are made for uniform flow conditions.

This idea of estimating Manning's "n" from field measurements of depth and discharge can be used for non-uniform flows conditions in irregular channels. The Mannings "n" for the channel may be determined by a trial and error procedure such that the difference between simulated steady, nonuniform water surface profile using standard step method (Chow,1959 and Subramanya,1984) and observed water surface profile is minimum. The above problem can be better formulated considering the following two aspects. Firstly,a formal search technique (Rao,1984) may be used which avoids trial and error,lends formality to calibration procedure, decreases arbitariness and gives likely error in the estimated value. Secondly, the best solution can be expected when abundance of data exist, which fortunately is the case with unsteady flow. In the present study a formal approach is developed for estimating the Manning's "n" from measured transient flow data. The developed methodology is based on non-linear optimization technique with the governing flow equations embedded as constraints.

1.2 Literature Review

The determination of parameters for a given system input and output and subject to boundary conditions is called an inverse problem. It is possible to estimate Manning's "n" from field measurements of discharge and depth using Optimization techniques. Parameter estimation techniques using unsteady flow data are more popular because large number of data can be obtained from a limited number of gauging stations. Significant work has been reported in the area of ground water engineering relating to parameter estimation. Solution techniques for aquifer parameter identification include Quasi linearization used by Yeh and Tauxe (1971), Linear programming used by Klei-

necke(1971),Multiple objective Linear Programming used by Neumann(1973) and Marquaardt's non-linear estimation algorithm used by Garay *et al.* (1976). However, the literature relating to identification of parameters in unsteady open channel flows is sparse.

Becker and Yeh (1972) proposed a methodology for parameter estimation in unsteady open channel flow using influence coefficient approach. Friction slope and exponent of hydraulic radius in empirical friction slope relation were considered as the parameters to be identified. The methodology involves minimizing the objective function which is the sum of the squares of error between observed data and simulated numerical solution of the equations based on some estimate of the parameters. An initial estimate of parameters is first specified and then these parameters are modified based on the evaluated objective function and influence coefficient matrix at each iteration. Becker and Yeh (1973a) extended this model for identification of multiple reach channel parameters. The parameters estimated were bed slope and Manning's "n" at every reach. The numerical solution of governing equations in the above models was based on explicit finite difference scheme method which requires a very small computational time step value. Also, the influence coefficient algorithm becomes tedious when used for estimating a large number of parameters. Yeh and Becker (1973) used Simplex linear programming algorithm with influence coefficient approach for estimating the unsteady open channel flow parameters,Manning's "n", exponent of hydraulic radius and the lateral inflow. The minmax criterion (minimizing the maximum of absolute values of errors) was taken as the objective function.

Fread and Smith (1978) proposed a methodology for estimating the parameter "n" as a function of stage or discharge and applied the methodology to a hypothetical ideal single reach of a channel. This method uses modified Newton Raphson algorithm to improve the initial trial values, such that the difference between observed and computed stages is minimum. The computed

stages are obtained from a four point implicit finite difference solution of unsteady one dimensional open channel flow equations subject to upstream and downstream boundary conditions of observed discharge and stage hydrograph respectively. Fread and Smith (1978) applied the above algorithm sequentially to a multiple reach river system by specifying the computed discharge at the downstream end of m^{th} reach as the upstream boundary condition for $(m+1)^{th}$ reach. The model proposed by Fread and Smith is not general in the sense that it can be applied to only dendritic river systems and stage observations at m stations are absolutely necessary to estimate the "n" values for m reaches. Moreover, numerical errors propagate easily in any sequential algorithm. Therefore, any errors of estimation in the upstream reaches may significantly affect the estimated values in the downstream reaches.

Recently, Wasantha Lal (1995) used Singular Value Decomposition method to estimate the Mannings roughness coefficient in one dimensional unsteady flow model. The method is used to solve for parameters after formulating the calibration problem as a generalized linear inverse problem. The calibration was repeated with different groups to determine variation of output error and uncertainty of parameters with parameter dimension. This method also uses influence coefficient approach for predicting the values of parameters at next iteration. But it solves underdetermined ,determined ,over determined and mixed dete- rmined problems.

In all these previous works, the parameters it is required to solve the governing equations in a separate solver outside the optimization model or algorithm.This may necessitate running of the solver a large number of times iteratively in order to determine the objective function value and the influence coefficient matrix. In the iterative method, when the simulation is outside the optimization model, convergence to an optimum may prove to be difficult if a large number of parameters are to identified. The aim of the present study is to estimate the parameter "n" by directly embedding the finite difference

approximations of the system of governing partial differential equations for unsteady open channel flow into the non-linear optimization model. This study involves an extension of the methodology proposed by A. Narayana(1996). A non-linear optimization technique is used since the momentum equation and the objective function are non-linear. Priessmann implicit scheme (Chaudhry,1993) is used for discretization of the governing equations to formulate the constraints. This helps in taking large computational time steps. Moreover, stability and convergence criteria need not be checked. The main advantage of using an embedded optimization model for solving the inverse problem is that an iterative procedure is not necessary and the flow hydraulics is simulated within the optimization model. Also, the embedding approach results in an elegant formulation and has the potential of incorporating very complex flow systems and boundary conditions.

1.3 Objective of the Study

Objectives of the present study are :

- To evaluate the proposed methodology for a number of specified channel systems including a single reach channel, a multiple reach channel and a multiple channel system;
- To study the effect of discretizing the governing equations on the performance of optimizer; and
- To evaluate performance of the proposed methodology for different scenarios of data availability and noise in the flow measurement data.

1.4 Organization of the Thesis

This thesis is organized as follows :

- Chapter 1 contains the introduction to this study and discussion to relevant literature;
- Chapter 2 presents the methodology and also discusses the simulation and Optimization algorithms used;
- Solution results and performance evaluations are discussed in Chapter 3, and
- Summary and Conclusions are presented in Chapter 4.

Chapter 2

Methodology

2.1 Methodology

The proposed model for estimating "n" involves the use of a nonlinear optimization technique with the finite difference form of the governing equations of flow forming the equality constraints. Governing equations are a set of nonlinear hyperbolic partial differential equations representing the conservation of mass and momentum. These equations are discretized using the Priessmann implicit finite difference scheme(Chaudhry, 1993). This scheme is used in the present study because large computational time steps can be considered. The same scheme is applied to the simulation model but the momentum equation is linearized. The optimization algorithm used in this study is the Sequential Quadratic Programming (SQP) which is available in the NAG library (NAG,1990) as the subroutine E04UCF. The routine is designed to minimize an arbitary smooth nonlinear objective function subject to constraints which may be simple bounds on variables, linear constraints and nonlinear constraints. The details of the methodology are presented in this chapter.

2.2 Governing Equations

In this study, one dimensional shallow water flow equations for a prismatic, wide rectangular channel are used as governing equations. However, it is simple to extend the methodology to flows in irregular channels. The governing equations represent the mass and momentum conservation and are written as (Chaudhry 1993) as follows.
Continuity equation:

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{2.1}$$

Momentum equation:

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}(q^2/h + gh^2/2) = gh(s_o - s_f) \tag{2.2}$$

where $h(x,t)$ = flow depth (m), $q(x,t)$ =discharge per unit width of channel(m^2/s), s_o =bottom slope of channel, s_f =friction slope, x=distance along flow direction(m),and t =time(s).
The friction slope, s_f is given by the Mannings equation:

$$s_f = \frac{q^2 n^2}{h^{10/3}} \tag{2.3}$$

where, n is the Manning's roughness coefficient

2.3 Simulation Model

A simulation model is used in the present study to simulate the observed flow data, in the absence of flow measurement data. The governing equations are linearized in the simulator. Momentum equation in nonconservation form along with the continuity equation(2.1) is used for the purpose.

$$\frac{\partial q}{\partial t} - \frac{q}{h}\frac{\partial h}{\partial t} + \frac{q}{h}\frac{\partial q}{\partial x} - \frac{q^2}{h^2}\frac{\partial h}{\partial x} - ghs_o + gh\frac{\partial h}{\partial x} - \frac{q|q|h}{h^2} = 0 \tag{2.4}$$

where k is the conveyance. The derivatives are now discretized using the Priessmann scheme. The variables and their time derivatives and space derivatives are discretized as given below.

$$\frac{\partial f}{\partial t} = \frac{(f_j^{k+1} + f_{j+1}^{k+1}) - (f_j^k + f_{j+1}^k)}{2\Delta t} \tag{2.5}$$

$$\frac{\partial f}{\partial x} = \frac{\theta(f_{j+1}^{k+1} - f_j^{k+1})}{\Delta x} + \frac{(1-\theta)(f_{j+1}^k - f_j^k)}{\Delta x} \tag{2.6}$$

$$f = \frac{1}{2}\left(\theta(f_{j+1}^{k+1} + f_j^{k+1}) + (1-\theta)(f_{j+1}^k + f_j^k)\right) \tag{2.7}$$

where, Δx = Grid spacing, Δt = Computational time step, θ = weighting coefficient, the subscript j refers to the grid point in space and the superscript k refers to the time level. In Eqs.(2.5) and (2.6), f refers to both q and h. In equation (2.7) f refers to variables s_f, h and q. The scheme is stable if $0.5 \leq \theta \leq 1.0$. In our study, $\theta = 0.67$ is used for single reach channel problem and $\theta = 0.8$ for multiple reach problems. Substituting the Eqs.(2.5)-(2.7) in Eqs.(2.1) and (2.4), Linearizing the equations by neglecting the second order terms in the power series expansion, and applying $f_j^{k+1} = f_j^k + \Delta f$ one can obtain continuity and momentum equations in the following form:

$$H_j \Delta h_{j+1} + b_j \Delta q_{j+1} = C_j \Delta h_j + D_j \Delta q_j + G_j \tag{2.8}$$

$$H_j' \Delta h_{j+1} + b_j' \Delta q_{j+1} = C_j' \Delta h_j + D_j' \Delta q_j + G_j' \tag{2.9}$$

The Eqs.(2.8) and (2.9) form a set of linear equations in the unknowns Δh and Δq at all the nodes $j = 1$ to N (N = Maximum number of nodes). The values of $H_j, b_j, C_j, D_j, G_j, H_j', b_j', C_j', D_j'$ and G_j' can be computed from the values of q and h at the known time level k. These values of values are known either from a previous computation or from specified initial conditions. Equations(2.8) and (2.9) along with the linearized form of boundary conditions are solved

simultaneously to obtain Δh and Δq at all the nodes $j = 1$ to N. Matrix inversion methods can be applied for solving the Equations(2.8) and (2.9). However, an efficient Double Sweep Algorithm as devised by Priessmann and Cunje is used for simulating observed data in case of single channels. In case of multiple channel systems, this method does not work and hence matrix inversion is used. This method works on the fact that if there exists a linear relationship of the form

$$\Delta q_j = E_j \Delta h_j + F_j \tag{2.10}$$

at a node j, then an analogous linear relationship

$$\Delta q_{j+1} = E_{j+1} \Delta h_{j+1} + F_{j+1} \tag{2.11}$$

also exists at point j+1. By substituting the value of Δq_j from Eq.(2.10) into Eq.(2.8) we get,

$$\Delta h_j = L_j \Delta h_{j+1} + M_j \Delta q_{j+1} + N_j \tag{2.12}$$

where,

$$L_j = \frac{H_j}{C_j + D_j E_j}; M_j = \frac{b_j}{C_j + D_j E_j}; N_j = -\frac{G_j + D_j F_j}{C_j + D_j E_j} \tag{2.13}$$

Substituting the value of Δq_j from Eq.(2.10) and Δh_j from Eq.(2.12) in Eq.(2.9), gives the following relation between Δq_{j+1} and Δh_{j+1} can be obtained.

$$\begin{aligned} \Delta q_{j+1} = & \frac{H_j(C'_j + D'_j E_j) - H'_j(C_j + D_j E_j)}{b_j(C_j + D_j E_j) - b_j(C'_j + D'_j E_j)} \Delta h_{j+1} \\ + & \frac{(G'_j + D'_j)(C_j + D_j E_j) - (G_j + D_j F_j)(C'_j + D'_j E_j)}{b'_j(C_j + D_j E_j) - b_j(C'_j + D'_j E_j)} \end{aligned} \tag{2.14}$$

which is of the form of Eq.(2.11). All the coefficients of Eqs.(2.8) and (2.9) can be evaluated, as they are in terms of flow variables at known time level. In this study an inflow hydrograph of the form $q_1^{k+1} = fu(t)$ is applied is applied as the

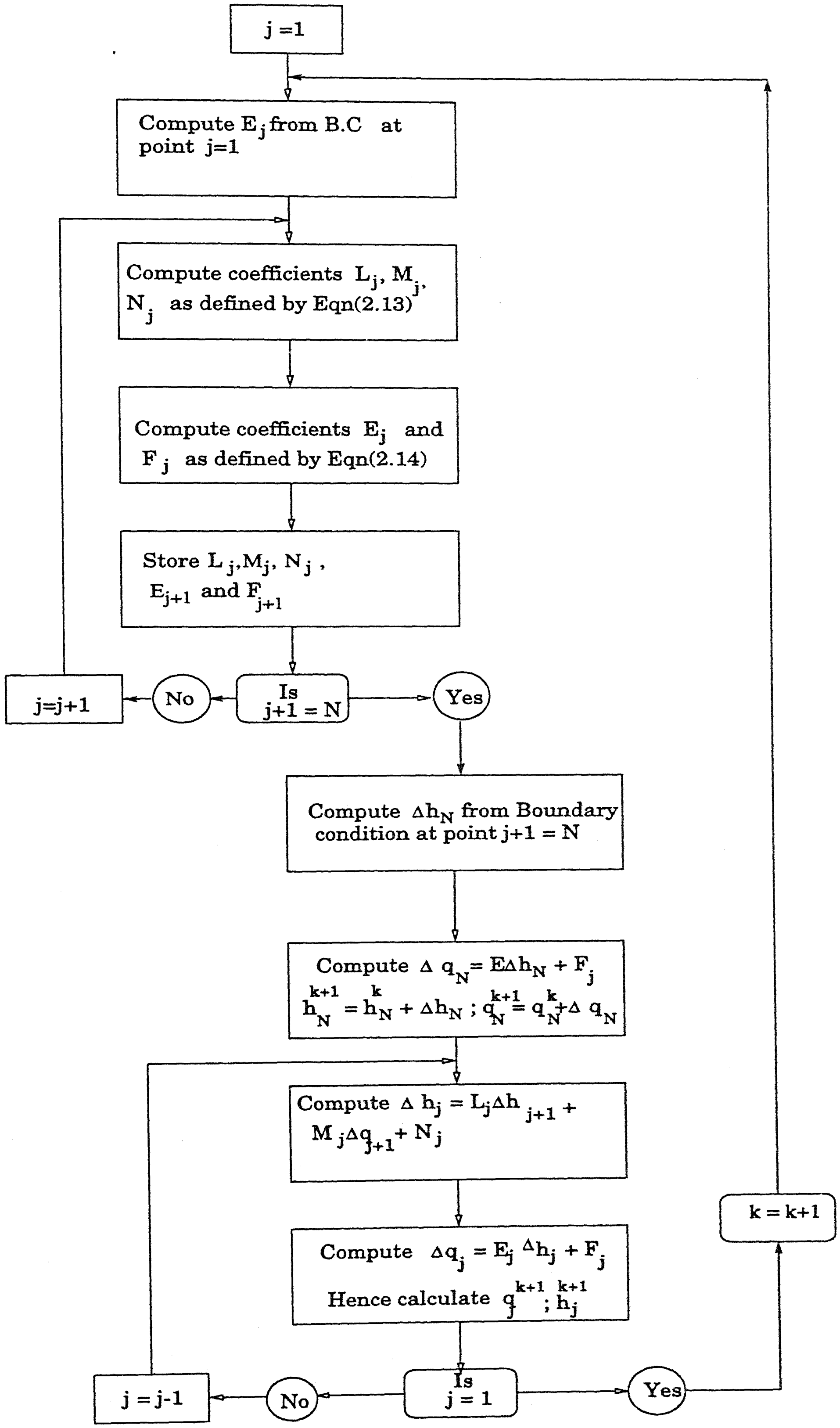

Figure 2.1: Flow Chart for Double Sweep Method of Solution

boundary condition at the upstream point. Here $fu(t)$ is the specified inflow hydrograph. Therefore, E_1 becomes 0 and $F_1 = \Delta q$. Uniform flow condition is used as the boundary condition at the downstream end. The linearized form of this boundary condition is given as, $\Delta q = (5/3)\frac{1}{n}(h_N)^{2/3}\Delta h\sqrt{s_o}$. Starting from upstream end, the values of E_j and F_j at all the grid points upto the last point is obtained. From the relations at the downstream end the value of Δh and hence h at next time level can be obtained. The value of q at next time level can be obtained from the Eq.(2.10). A backward procedure is adopted to get the values of flow variables at all the upstream nodes at the next time level. The above procedure is summarized in the Fig. 2.1.

2.4 Formulation of the Inverse Problem

The inverse problem is formulated as a non-linear optimization model with roughness coefficient (n) as an unknown variable. The objective function is defined as the sum of squares of the differences between the estimated and measured values of flow depth and/or discharge. Finite difference analogs of the governing Eqs.(2.1) and (2.2) , and specified boundary conditions at the upstream and downstream ends constitute the equality constraints of the optimization model. The continuity equation for each node with the finite difference node together with the upstream boundary condition at the first node form the linear equality constraints. The momentum equation for each node along with the downstream boundary condition form the non-linear equality constraints. Formulation of the inverse problem for a single reach channel can be summarised as follows.

2.4.1 Objective function

Minimize :

$$\sum [(x(o) - x(m))/x(m)]^2 \qquad (2.15)$$

Where, x(o) are the simulated depth or discharge values at an observation station and $x(m)$ are the observed depth or discharge values at the corresponding points.

2.4.2 Linear constraints

Finite difference analog of the continuity equation using the Priessmann scheme gives the linear constraints in the optimization model.

$$\frac{(h_{i+1}^{k+1} + h_i^{k+1}) - (h_{i+1}^k + h_i^k)}{2\Delta t} + \theta \frac{(q_{i+1}^{k+1} - q_i^{k+1})}{\Delta x} + (1-\theta)\frac{(q_{i+1}^k - q_i^k)}{\Delta x} = 0 \quad (2.16)$$

for i = 1 to N-1 ; k = 1 to p

where N = total number of grid points , p = total number of time steps in the computational domain and $\theta = 0.67$. The specified upstream boundary condition is also included as a linear constraint. This is given as

$$q_i^{k+1} = fu(t) \qquad (2.17)$$

where, fu(t) = Specified inflow hydrograph.

2.4.3 Non linear constraints

Finite difference analog of the momentum equation using the Priessmann scheme forms the set of nonlinear constraints.

$$\frac{(q_{i+1}^{k+1} + q_i^{k+1}) - (q_{i+1}^k + q_i^k)}{2\Delta t} + (\theta/\Delta x)\left(\frac{(q_{i+1}^{k+1})^2}{h_{i+1}^{k+1}} - \frac{(q_i^{k+1})^2}{h_i^{k+1}}\right) + ((1-\theta)/\Delta x)$$

$$\left(\frac{(q_{i+1}^k)^2}{h_{i+1}^k} - \frac{(q_i^k)^2}{h_i^k}\right) + (g\theta/2\Delta x)((h_{i+1}^{k+1})^2 - (h_i^{k+1})^2) + (g(1-\theta)/2\Delta x)((h_{i+1}^k)^2 - (h_i^k)^2)$$

$$+(g\theta n^2/2)\left(\frac{(q_{i+1}^{k+1})^2}{(h_{i+1}^{k+1})^{7/3}} + \frac{(q_i^{k+1})^2}{(h_i^{k+1})^{7/3}}\right) - gs_o(\theta/2(h_{i+1}^{k+1} + h_i^k) + (1-\theta)/2(h_{i+1}^k + h_i^k))$$

$$+g((1-\theta)n^2/2)\left(\frac{(q_{i+1}^k)^2}{(h_{i+1}^k)^{7/3}} + \frac{(q_i^k)^2}{(h_i^k)^{7/3}}\right) = 0 \qquad (2.18)$$

for i = 1 to N-1 ; k = 1to p.

A rating curve if available, can be specified as the boundary condition at the downstream end. In all the studies reported here , a uniform flow condition is assumed as the downstream boundary condition. This is given as

$$q_N^{k+1} = \frac{(h_N^{k+1})^{5/3}\sqrt{s_o}}{n} \qquad (2.19)$$

Eq.(2.19) is also included in the set of nonlinear constraints. Formulation of the parameter estimation model for multiple reach system is exactly same as the prior described model for a single reach channel, except for the fact that the number of decision variables will increase as the number of "n" values to be estimated are multiple. Formulation of the parameter estimation model for a channel system is very similar to the prior described model for a single reach channel with the compatibility conditions at junctions forming additional equality constraints. Mass balance and energy balance conditions at the junctions constitute these compatibility conditions. It should be noted here that other inequality constraints which incorporate lower and upper bounds on parameters can be also included in the optimization model. The nonlinear optimization problem described in this section is solved using the Sequential Quadratic Programming (SQP) algorithm (Powell 1974) as coded in the NAG library (Numerical Algorithm Group , 1990).

2.4.4 Optimization algorithm

The nonlinear optimization model is solved using the sequential quadratic programming(SQP) algorithm, as coded in the NAG library(NAG,1990). The salient features of this algorithm as described in the NAG Users Manual is presented here.

The SQP algorithm is used to solve the general optimization problem which can be stated as follows :

$$\textbf{Min :}F(x)\ \textbf{Subject to}\ l \leq \left\{ \begin{array}{c} x \\ A_L \\ C(x) \end{array} \right\} \leq u \tag{2.20}$$

where F(x) is the nonlinear objective function, A_L is an n_L x n coefficient matrix, C(x) is an n_N element vector of nonlinear constraints. The objective function and constraint functions are assumed to be smooth i.e., at least twice-continuosly differentiable. Here upper and lower bounds are specified for all variables and constraints. An equality constraint can be specified by setting $l_i = u_i$. If certain bounds are not present, the associated elements of l or u can be set to special values that will be treated as $\pm\infty$. The initial estimates of the solution to (2.20), together with subroutines that define $F(x)$, $c(x)$ and as many first partial derivatives as possible must be specified; unspecified derivatives will be approximated by finite difference at a non-trivial expense. In this study, all the first partial derivatives of nonlinear constraints and objective function are specified by the program in the form of Jacobian and Objective gradient, respectively. Since the initial estimates may not be close to actual ones Cold Start option is used.

At the solution of Eq.(2.20), some of the constraints will be active, i.e., satisfied exactly. An active simple bound constraint implies that the corresponding variable is fixed at its bound, and hence the variables are partitioned into fixed and free variables. A point x is a *first-order Kuhn-Tucker point* for

Eq.(2.20) if the following conditions hold(Powell, 1974):

1. x is feasible;

2. there exists vectors ξ and λ (the Langrange multiplier vectors for bound and general constraints) such that

$$g = C^T\lambda + \xi \tag{2.21}$$

where g is the gradient of F evaluated at x, and $\xi_j = 0$ if the j^{th} variable is free.

3. The Langrange multiplier corresponding to an equality constraint active at its lower bound must be non-negative, and non-positive for an inequality constraint active at its upper bound.

There are several ways of organizing the matrix calculations of active set algorithm for quadratic programming. By working with the upper triangular matrix method for quadratic programming, it is possible to carry out matrix operations economically. The advantage of this approach is that there is substantial saving of computer storage, because the orthogonal matrix is not required. All these features are included in the E04UCF subroutine(NAG,1990), which is used in this study. The basic structure of E04UCF involves major and minor iterations. The major iterations generate sequence of x_k that converge to x^*, a first order Kuhn-Tucker Point of Eq.(2.20). At a typical major iterations, the new iterate $\overline{x}$ is defined by

$$\overline{x} = x + \alpha p \tag{2.22}$$

where x is the current iterete, the non-negative scalar α is the step length and p is the search direction. Also associated with each major iteration are estimates of Langrange multipliers and prediction of active set. The search direction p is the solution of the quadratic programming subproblem of the form

$$\text{Minimize: } g^T p + \frac{1}{2} p^T H p$$

$$\text{subject to } \bar{l} \leq \left\{ \begin{array}{c} p \\ A_{L}p \\ A_{N}p \end{array} \right\} \leq \overline{u} \tag{2.23}$$

where g is the projected gradient of aF at x, the matrix H is the positive definite quasi-Newton approximation to the Hessian of the Langrangian function and A_N is the jacobian matrix of c evaluated at x. Let l in Eq.(2.20) be partitioned into three sections: l_B,l_L and l_N, corresponding to bound, linear and nonlinear constraints. The vector $\bar{l}$ is similarly partitioned and defined as $\bar{l}_B = l_B - x, \bar{l}_L = l_L - A_L x$, and $\bar{l}_N = l_N - c$,
where c is the vector of nonlinear constraints evaluated at x. The vector $\overline{u}$ is defined in an analogous fashion.

The estimated Langrangian multipliers at each iteration are Langrange multipliers from subproblem (2.23). Since solving a quadratic subprogram is an iterative procedure, the minor iterations of the main problem form the iterations of the quadratic programming subproblem.

In general, a quadratic program must be solved by iteration . Let p denote the current estimate of the solution of (2.23); the new iterate $\overline{p}$ is defined by

$$\overline{p} = p + \sigma d \tag{2.24}$$

where, as in (2.22), σ is the non-negative step length and d is a search direction.

At the beginning of each iteration of E04UCF, a working set is defined of constraints (general and bound) that are satisfied exactly. The vector d is then constructed so that the values of constraints in the working set remain unaltered for any move along d. For bound constraint in the working set this property is achieved by setting the corresponding component of d to zero, i.e. by fixing the variable at its bound. The subscripts 'FX' and 'FR' denote the selection of components associated with fixed and free variables. The general constraints in the working set will remain unaltered if

$$C_{FR} d_{FR} = 0, \tag{2.25}$$

which implies

$$d_{FR} = Zd_z \tag{2.26}$$

some vector d_z,C is the submatrix of rows of $\begin{pmatrix} A_L \\ A_N \end{pmatrix}$ corresponding to general constraints in the working set and Z is the matrix associated with the TQ factorisation of C_{FR}. Each change in working set leads to a simple change in C_{FR}. A row or a column of C_{FR} changes if the status of a general constraint or bound constraint are altered respectively.

After obtaining the search direction from the Quadratic programming Subproblem, each major iteration procedes by determining a step length α which leads to a sufficient decrease in the Langrangian merit function given as under :

$$L(x, \lambda, s) = F(x) - \sum_i \lambda_i \left(c_i(x) - s_i\right) + \frac{1}{2} \sum_i \rho_i \left(\left(c_i(x) - s_i\right)^2\right. \tag{2.27}$$

where x, λ and s vary during linesearch. The summation terms in the above function involve only the nonlinear constraints. The vector λ is an estimate of the Langrange multipliers for the nonlinear constraints of Eq. (2.20) and s_i are the non-negative slack variables for nonlinear inequality constraints. The solution of QP subproblem (2.23) provides a vector triple that serves as the search direction for the three sets of the variables.

At the end of each major iteration, a new Hessian approximation $\overline{H}$ is defined as a rank-two modification of H.

$$\overline{H} = H - \frac{1}{s^T H s} hss^T H + \frac{1}{y^T s} yy^T \tag{2.28}$$

where $s = \overline{x} - x$ (change in x) In E04UCF, H is required to be positive definite. If H is positive definite, $\overline{H}$ difined by Eq(2.28) will be positive-definite if and only if y_T is positive. Ideally, y in Eq(2.28) would be taken as y_L, the change in gradient of the Langrangian function

$$y_L = \overline{g} - \overline{A}_N^T \mu_N - A_N^T \mu_N \tag{2.29}$$

where μ_N denotes the QP multipliers associated with the nonlinear constraints of the original problem. If y_L^T is not suficiently positive, an attempt is made to perform the original update with vector of the form

$$y = y_L + \sum_{i=1}^{m_N} \omega_i (a_i \overline{x} c_i(x) - a_i(x) c_i(x)) \tag{2.30}$$

where $\omega_i \geq 0$. if no such vector can be found, the update is performed with scaled y_L.

To summarise, each major iteration the routine performs before reaching an optimal point or otherwise includes,

1. the solution of the quadratic programming subproblem,
2. a line search with an augmented Langrangian merit function and
3. a quasi-Newton update of the approximate Hessian of the Langrangian function.

Chapter 3

Results and Discussion

3.1 Performance Evaluation

Performance of the proposed optimization model for the estimation of roughness coefficient is evaluated using illustrative example problems for hypothetical open channel systems. These example problems include (1) flow in a single channel with a single value of "n" (2) a single straight channel with multiple reaches and multiple "n" values and (3) a simple dendritic system of three channels with a multiple roughness values corresponding to different reaches. The observation data for these cases are simulated by the solving governing Eqns. (2.1) and (2.2) for assumed true values of n using the Priessmann scheme. These simulated observation data for discharge and flow depth are then used in the optimization model to estimate the roughness coefficients. Identical initial and boundary conditions are applied while obtaining the simulated observation data and while solving the optimization model. The advantage of using simulated observation data is that it provides a means to delineate the actual deviation between the true values and the estimated values of "n". In this approach the simulated measurement data are free of measurement noises.

3.1.1 Single channel problem

A wide rectangular channel with a constant bed slope of 0.0004 is considered in this case. The length of the channel is 40,000m. A uniform flow depth,h = 2.5m and corresponding uniform flow discharge for n = 0.023 are specified as the initial steady state conditions. A Log-Pearson Type III hydrograph as shown in Fig. 3.1 is specified as the upstream boundary condition. In this hydrograph, q_b = initial discharge=$3.32m^2/s$, $q_p = 6.0m^2/s$, t_p = time to peak = 2hr, t_g = time to centroid of hydrograph = 2.5hr and t_b = time base of hydrograph = 6hr. Variation of q with time for this hydrograph is given by the following equation

$$q = q_b + (q_p - q_b)e^{\frac{-(t-t_p)}{(t_g-t_p)}}(t/t_p)^{\frac{t_p}{(t_g-t_p)}} \tag{3.1}$$

Flow measurement data for this case are simulated by numerically solving the governing equations (2.1) and (2.2) using the Priessmann scheme. These measurement data for this case are obtained at half hour intervals ($\Delta t = 0.5$ hr) at measurement locations thousand metres apart ($\Delta x = 1000$ m). These simulated observation data for q and h without any observation noise are used in the optimization model to estimate the roughness coefficient. A larger grid size is considered in the optimization model for reducing the computational costs even though measurement data are available at smaller grid sizes. Manning's roughness coefficient, "n" is estimated for different scenarios as presented in Table 3.1 in order to study the grid size effect on the parameter estimation . The initial estimate of "n" which is required to start the optimization model is specified to be 0.01 in all the runs. It can be readily seen from Table 3.1 that the optimization model performs very well for the case of a single channel even if Δx and Δt are large. The maximum difference between the estimated value and the true value is only 0.0001 . An average of 10 iterations of the Optimization model are required to arrive at these optimal "n" values. Estimated

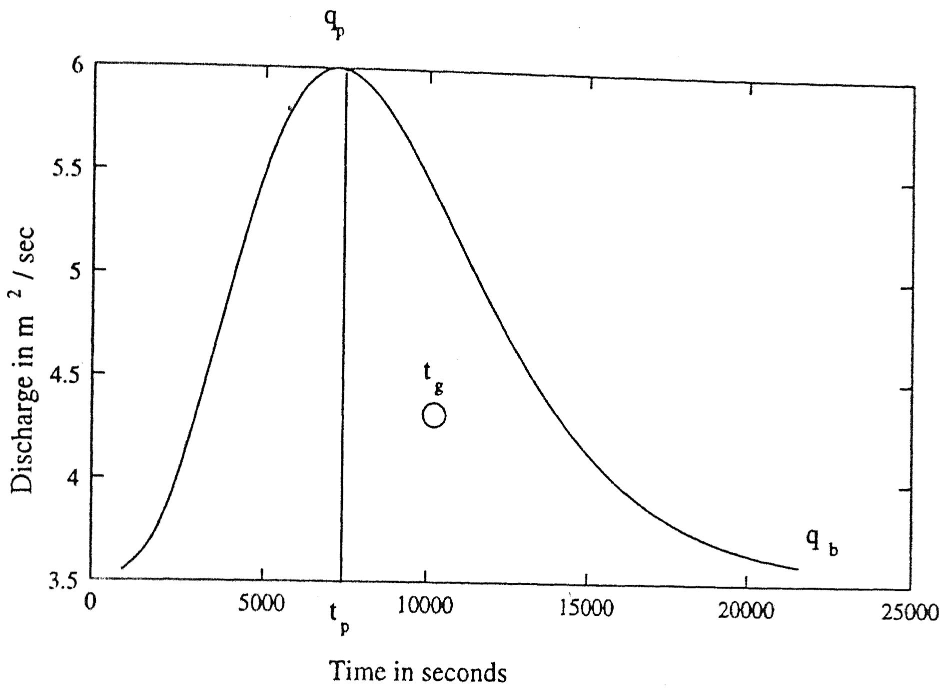

Figure 3.1: Log Pearson TypeIII Hydrograph

and observed stage hydrographs at x = 20 km for Scenario 1 are compared in Fig. 3.2. This comparision indicates satisfactory performance of the optimization model. The Scenario 1 has 253 variables, 126 linear and non- linear constraints each, respectively. So, the CPU time for this problem is 3min and 12sec. When the grid spacing is increased, as in other Scenarios of Table 3.1 the CPU time reduces and it comes down to a few seconds for Scenario 3.

The effect of different initial estimates of "n" on the obtained optimum solution for "n" is shown in Table 3.2. Results shown in Table 3.2 are obtained using $\Delta x = 2000$ m and $\Delta t = 1$ hr in the optimization model. It can be seen from Table 3.2 that the optimization model performs satisfactorily even when the initial estimate is far off from the true value. However, as may be expected, less number of iterations are required when the initial estimate is close to the true value. In all the cases, since the order magnitude of flow variables are known at all points from simulated observation data, the intial estimates for

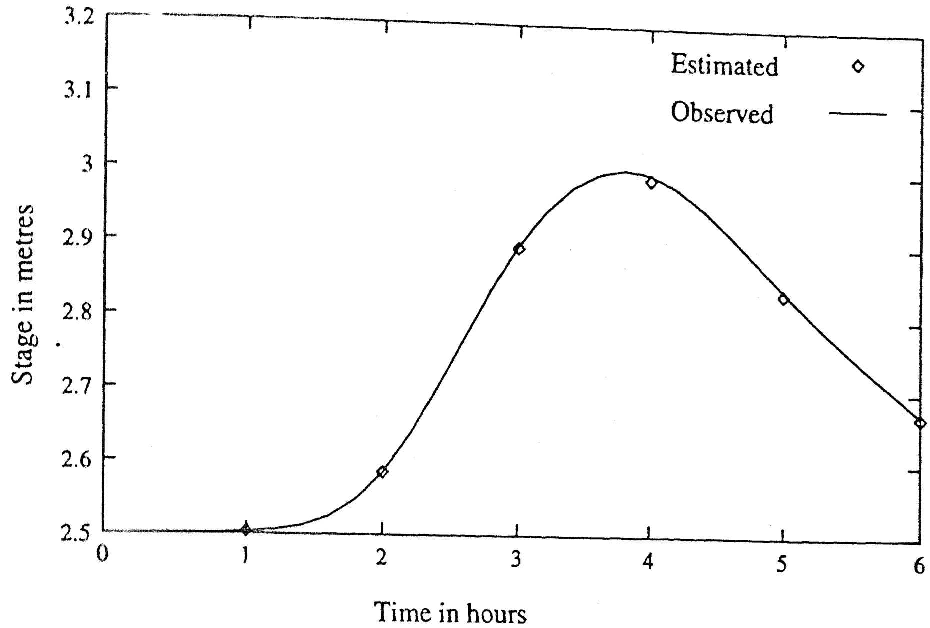

Figure 3.2: Estimated and observed stage hydrographs at X= 20 km (Scenario 1)

Table 3.1: Grid size effect on the parameter estimation

Scenario no.	Grid size in the Optimizer		Estimated Value of "n"	No. of iterations	Objective function
	Δx (m)	Δt (hr)			
1	2000	1.0	0.0229	12	0.236
2	4000	1.0	0.0229	9	0.118
3	8000	1.0	0.0229	9	0.053
4	4000	0.5	0.0228	9	0.04
5	8000	0.5	0.0228	9	0.044

flow variables are taken as equal to the observed values. The solution results are identical even the initial estimates for the flow variables are 80% of observed values .

Table 3.2: Effect of initial estimate on the performance of the model

Scenario no.	Initial estimate of "n"	Optimum value of "n"	No. of Iterations
6	0.2	0.0233	42
7	0.015	0.0229	7
8	0.005	0.0229	15

3.1.2 Multiple reach problem

3.1.2.1 single channel

A straight channel consisting of six reaches of equal length, each 6000m and having different roughness coefficients is considered in this case. All the reaches have identical bottom slopes, $s_o = 0.0004$. The flow measurement data for this system is simulated using the inflow hydrograph shown in Fig. 3.1 as the upstream boundary condition. The true values of "n" used in these simulations are 0.018, 0.023 0.028, 0.025, 0.031 and 0.035, respectively. The observation data is simulated using $\Delta x = 1000$ m, $\Delta t = 0.5$hr and $\theta = 0.67$ in the simulator. The initial steady state conditions are obtained by solving the gradually varied flow equations corresponding to a discharge of $3.53 m^2/s$ and a downstream depth of 2.98m. The simulated flow measurement data are used in the objective function of the optimization model to estimate the "n" values for all the six reaches. The initial estimates for all the six "n" values are specified to be 0.022. Table 3.3 shows the effect of grid size on performance of the optimization model. It can be observed from Table 3.3 that the optimization model performs satisfactorily even in the case of multiple reach problem. As may be expected, errors in the estimated values of n increase as Δt and Δx are increased. For example, the maximum error in the estimation is only 0.0017

for the scenario 9. Estimated and observed stage hydrograph at $x = 18$ km are compared in the Fig. 3.3. The CPU time required for Scenario No.9 is around 4 min and decreases as the number of variables decease. An average of 15 iterations are required to arrive at these optimal values.

Table 3.3: Grid size effect:Multiple reach single channel problem

Scenario no.	Grid Size		Estimated Value of "n"					
	Δx (m)	Δt (hr)	Reach no.					
			1	2	3	4	5	6
9	2000	1.0	0.0178	0.0220	0.0263	0.0244	0.0298	0.0352
10	3000	1.0	0.0176	0.0212	0.0245	0.0232	0.0280	0.0348
11	2000	2.0	0.0176	0.0221	0.0274	0.0254	0.0311	0.0369

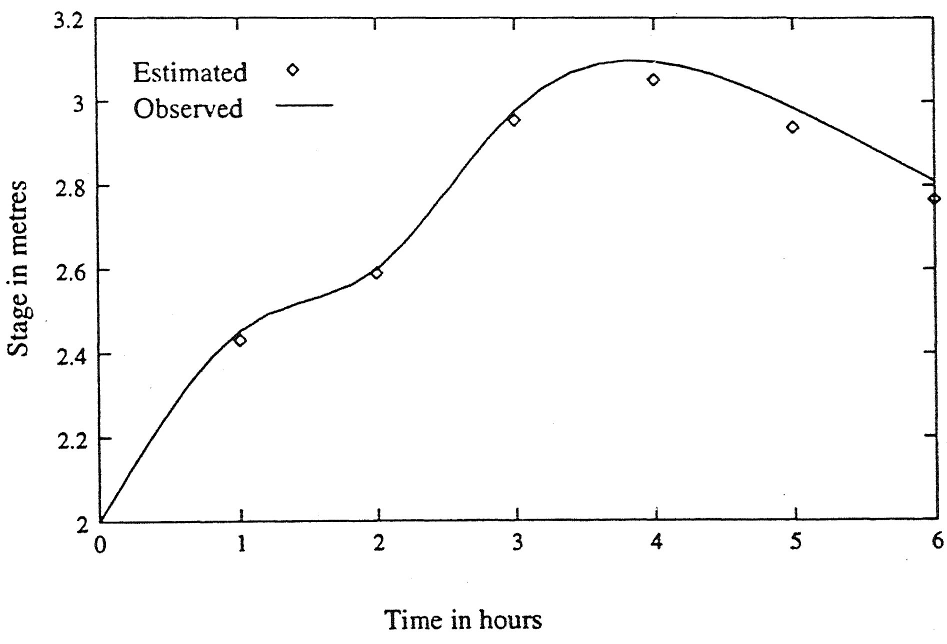

Figure 3.3: Estimated and observed stage hydrographs at the end of third reach (Scenario 9)

3.1.2.2 Multiple channel system

A dendritic system of three wide rectangular channels as shown in Fig. 3.4 is considered in this case. Each channel is divided into two reaches of equal length, with each reach having a different roughness coefficient. All three channels are of the same length = 12,000m, with identical bed slope, s_o = 0.0004. The flow measurement data for this system is simulated also by using the inflow hydrograph shown in Fig. 3.1 as the upstream boundary condition. The true values of "n" used in these simulations are 0.018, 0.023, 0.028, 0.025, 0.031 and 0.035 for the reaches 1,2,3,4,5 and 6, respectively. The observation data is simulated using $\Delta x = 1000$ m and $\Delta t = 0.5hr$ and $\theta = 0.8$ in the simulator. The initial steady state conditions for this system are obtained by solving the gradually varied flow equation corresponding to discharges of $3.53m^2/s$, $2.27m^2/s$ and $5.80m^2/s$ in channels 1,2 and 3 respectively. The initial downstream depth in channel 3 in these computations is 4.02 m.

The simulated flow measurement data are used in the objective function of the optimization model to estimate the "n" values for all the six reaches. The initial estimates for all the six "n" values are specified to be 0.022. Table 3.4 shows the effect of the grid size on the performance of the optimization model. It can be observed from Table 3.4 that the optimization model performs satisfactorily even in the case of multiple channel problem. As may be expected, errors in the estimated values of n increase as Δt and Δx are increased. For example, the maximum error in the estimation is only 0.0013 for the Scenario 12 ($\Delta x = 2000$ m and $\Delta t = 1$ hr) as compared to an error of 0.0025 for scenario 13 ($\Delta x = 6000$ m , $\Delta t = 1$ hr). Identical results are obtained when the initial estimates for "n" are taken equal to 0.01 indicating the robustness of the proposed model. Estimated and observed stage hydrographs 6000m downstream of the junction point, along the channel 3 for Scenario 12 are compared in Fig. 3.5. Satisfactory performance of the optimization

model can be observed again from this figure. Maximum difference between the estimated and the observed flow depths is only 0.11 m.

Table 3.4: Grid size effect:Multiple channel problem

Scenario no.	Grid Size		Estimated Value of "n"					
	Δx (m)	Δt (hr)	Reach no.					
			1	2	3	4	5	6
12	2000	1.0	0.0179	0.0217	0.0287	0.0253	0.0306	0.0364
13	6000	1.0	0.0183	0.0205	0.0279	0.0236	0.0306	0.0362
14	2000	2.0	0.018	0.0229	0.0273	0.0276	0.0318	0.0373

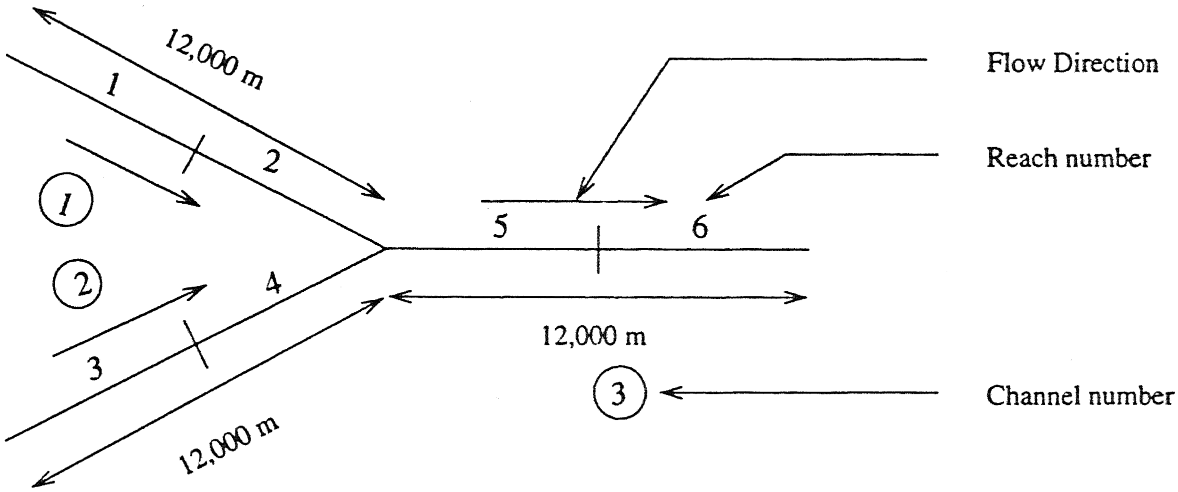

Figure 3.4: Schematic layout of the multiple channel system

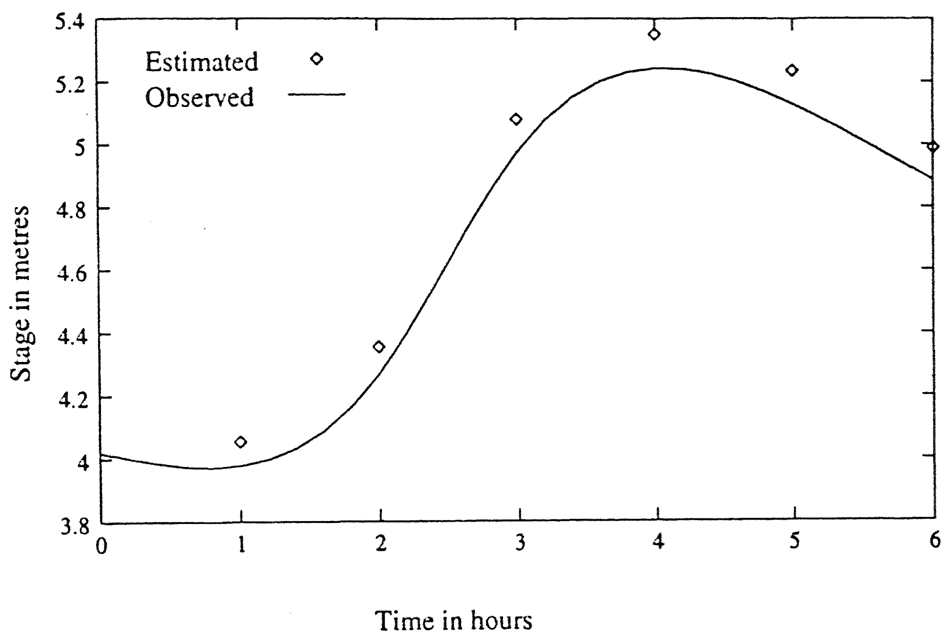

Figure 3.5: Estimated and observed stage hydrographs 6000m downstream of junction point in channel 3 (Scenario 12)

3.1.2.3 Performance evaluation of the model for missing data

In many cases, observation data may be available either for flow depths or discharges. Also, data may be available only at few selected stations and at few selected times. However, it may be required to use smaller Δx and Δt in the optimization model while estimating the parameters in order to reduce the discretization errors. Performance of the optimization model under such conditions is studied by considering several scenarios as presented in Table 3.5. Δx and Δt in all these studies are 2000m and 1hr, respectively. The initial estimates for the flow variables are taken as spatial mean of the available data at that time level. Initial estimates for the "n" values are taken as 0.022. Results presented in table 3.5 indicate that the proposed model performed satisfactorily even when the availability of the measurement data is sparse (Scenarios 15-18). However, there is a significant deterioration in the performance of the model when the flow depth measurements are not available (Scenario 19). The maximum error in estimation of "n" for this scenario being 0.010, in reach 5. Results are also not satisfactory, as expected, in the case of an underdetermined system (Scenario 20) i.e., the number of observation stations is less than the number of parameters to be estimated. The maximum error in estimation of "n" being 0.012 for this scenario in reach 5.

Table 3.5: Performance evaluation of the model for the multiple channel problem

Scenario no.	Availability of Observed Data	Estimated "n" Values
15	(i) both q and h (ii) t = 1,3 & 5 hrs (iii) All Spatial Grid Points	0.0178, 0.0206, 0.0296, 0.0246, 0.0299 and 0.0359
16	(i) both q and h (ii) t = 1 & 4 hrs (iii) All spatial Grid Points	0.0182, 0.021, 0.0293, 0.0249, 0.0307 and 0.0361
17	(i) both q and h (ii) all times (iii) only the two end points of a reach	0.018, 0.0206, 0.0291, 0.0217 0.0309 and 0.0365
18	(i) only h (ii) all times (iii) all spatial grid points	0.0183, 0.0236, 0.0278, 0.0256, 0.0336, 0.0384
19	(i) only q (ii) all times (iii) all spatial grid points	0.0203, 0.0191, 0.0367, 0.0238, 0.0211 and 0.0363
20	(i) both q and h (ii) all times (iii) only for longitudinal mid points of reaches 1,3 & 5 (under determined system)	0.0191, 0.0235, 0.0276,0.0197, 0.0323 and 0.0429

3.1.2.4 Performance evaluation of the model in the presence of observation noise

In many cases , flow measurement data from field conditions may contain observation noise. Performance of the proposed parameter estimation model in the presence of observation noise is evaluated here. In the illustrative case considered here , all the input data except the flow measurement data are identical to those for Scenario 12. Simulated noise free measurement data for Scenario 12 is modified by adding a random error term as follows

$$x'_i = x_i + \epsilon_i \tag{3.2}$$

where , x_i =simulated noise free measurement of ith data point, and ϵ_i = a random error term sampled from a uniform distribution of zero mean , and upper and lower limits of $+\alpha x_i$ and $-\alpha x_i$, respectively. The NAG routine G05FAF is used to generate the random numbers. This program generates a vector of pseudo random numbers uniformly distributed over an interval (a,b). In this evaluation , five sets of noisy flow measurement data are generated using the prior procedure. These data sets are then incorporated into the objective function of the parameter estimation model to estimate the roughness values. The "n" values obtained for every random number in cases of induced errors of 10% and 20% are given in Tables 3.6 and 3.7, respectively. The average of "n" values obtained as solutions for the five sets of noisy data with α = 0.1 and 0.2 are also presented. The estimated average values of n for both the cases of $\alpha = 0.1$ and 0.2 are close to those for $\alpha = 0.0$ representing, error free measurement. The sum of squares of the estimation errors is presented as a function of α in Fig. 3.6. Tables(3.6 and 3.7) and Fig. 3.6 indicate the potential applicability of the proposed model for real life situations where the measurement data are erroneous.

Table 3.6: Estimated values of "n" for an $\alpha = 0.1$

Erroneous measurement data set no.	Estimated Value of "n"					
	Reach no.					
	1	2	3	4	5	6
1	0.0177	0.0216	0.0307	0.0249	0.0296	0.0359
2	0.0174	0.0211	0.0286	0.0247	0.0313	0.0362
3	0.0170	0.0215	0.0284	0.0241	0.0301	0.0356
4	0.0181	0.0232	0.0308	0.0259	0.0302	0.0368
5	0.0194	0.0218	0.0291	0.0262	0.0313	0.0366
Mean values	0.0179	0.218	0.0295	0.0252	0.0306	0.0362
True values	0.018	0.023	0.028	0.025	0.031	0.035

Table 3.7: Estimated values of "n" for $\alpha = 0.2$

Erroneous measurement data set no.	Estimated Value of "n"					
	Reach no.					
	1	2	3	4	5	6
1	0.0173	0.0216	0.0332	0.0247	0.0281	0.0351
2	0.0169	0.0204	0.0285	0.0240	0.0322	0.0353
3	0.0160	0.0218	0.0280	0.0239	0.0296	0.0343
4	0.0182	0.0251	0.0328	0.0269	0.0295	0.0366
5	0.021	0.0218	0.0291	0.0275	0.0318	0.0363
Mean values	0.0177	0.220	0.0303	0.0254	0.0302	0.0355
True values	0.018	0.023	0.028	0.025	0.031	0.035

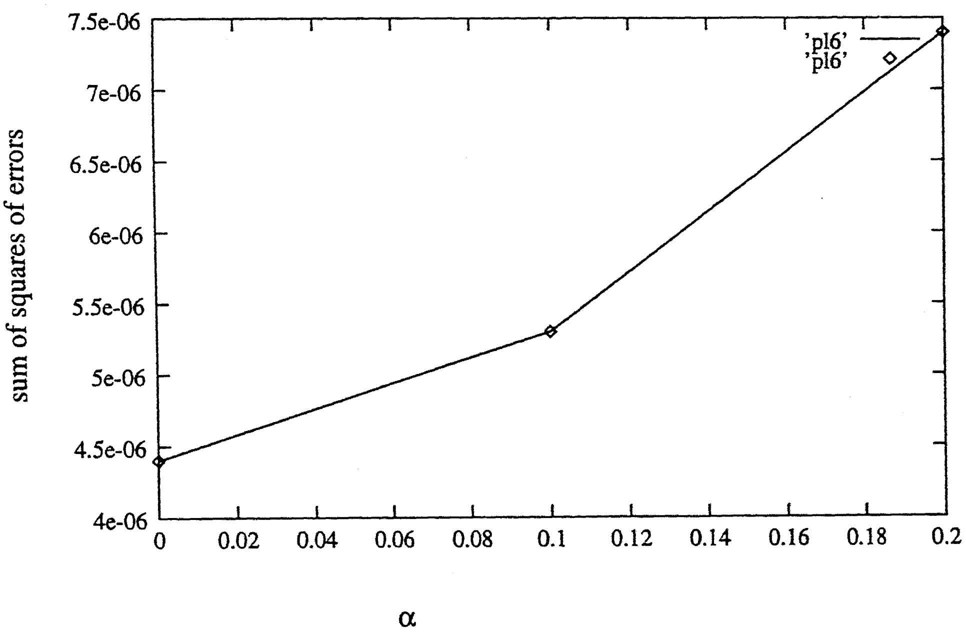

Figure 3.6: Effect of α on the parameter estimation

Chapter 4

Summary And Conclusion

In this study, an embedded optimization model is presented for estimating the Manning's roughness coefficients from unsteady flow measurement data in open-channel systems. Finite-difference approximations of the governing flow equations are embedded as equality constraints in a nonlinear optimization model. Priessmann finite difference method is used for discretizing the governing partial differential equations. The nonlinear objective function based on the sum of squares of error is minimized by using the Sequential Quadratic Programming (SQP) algorithm as coded by Numerical Algorithm Group (NAG). Performance of the proposed model is evaluated for different scenarios of data availability and observation noise in the flow measurement data. It is found that the model performs satisfactorily in all the cases except (1) when only discharge measurements are available and (2) when the number of observation stations is less than the number of parameters to be estimated. Effects of grid size and initial estimates for parameters on the model performance are also studied. Results of these evaluations show that the performance of the model is satisfactory even though global optimality of the solution cannot be guaranteed. Also, the applicability of the proposed approach to very large domain problems should be further explored, considering some of the inherent limitations of the embedding technique.

Bibliography

[1] Barnes, H.H. Jr.," Roughness Characteristics of Natural Channels" *"U.S. Geological Survey Water Supply paper 1849"*, Washington, 1967.

[2] Becker, L., and Yeh, W. W-G., Identification of parameters in Unsteady Open Channel Flows, *"Water Resources Research"*, Vol. 8, No. 4, Aug., 1972, pp. 956-965

[3] Becker, L., and Yeh, W. W-G., The Identification of Multiple reach channel parameters, *"Water Resources Research"*, Vol. 9, No. 2, Apr., 1973a, pp.326-335.

[4] Becker, L., and Yeh, W. W-G., Linear Programming and Channel Flow Identification, *"Journal of the Hydraulics Division,"* Vol. 99, No. HY11, Nov., 1973b, pp. 2013-2021.

[5] Chaudhry, M.H., *"Open Channel Flow,"* Prentice Hall of India, New Delhi, 1993.

[6] Chow, V.T., *"Open Channel Hydraulics,"* McGraw-Hill Book Company, New York, 1959

[7] Dennis, J.E., Jr. and Schnabel, R.B., *"A New Derivation of symmetric Positive Definite Secant Updates,"* In:'Nonlinear Programming 4', O.L. Mangasarian, R.R. Meyer and S.M. Robinson, (eds), Academic Press, London and Newyork, pp. 167-199, 1981

[8] Fread, D.L., and Smith, G.G., *"Calibration techniques for 1-D unsteady flow models,"* Journal of Hydraulic Division, Vol.104, No.7, Jul., 1978, pp. 1027-1043.

[9] French, R.H., *"Open Channel Flow,"* Chap. 4., Development of Uniform flow Concepts, McGraw-Hill Book Company, New York, 1985, pp. 111-162

[10] Garay, H.L., Haimes, Y.Y., and Das, P., "Distributed Parameter Identification for Ground Water Systems by nonlinear Estimation," *Journal of Hydrology* Vol. 30, No. 1/2, May., 1976, pp. 47-61.

[11] Kleinecke, D., Haimes, Y.Y., and Das, P., Use of Linear Programming for Geohydrologic Parameters of Groundwater Basins, *"Water Resources Research,"* Vol. 7, No. 2, Apr., 1971, pp. 367-374.

[12] Narayana, A., *"Optimal estimation of the Roughness coefficients using unsteady open channel flow data,"* M.Tech Thesis, Dept. of Civil Engineering, I.I.T. Kanpur, India, p.36, 1996.

[13] Neumann, S.P., Calibration of Distributed Parameter Ground Water Flow Models Viewed as a Multiple-Objective Decision Process under Uncertainity, *"Water Resources research,"* Vol. 9, No. 4, Aug., 1973 , pp. 1006-1021.

[14] Numerical Algorithm Group NAG *"E04UCF - NAG Fortran Library Routine Document,"* Mark 14, London, 1990

[15] Powell, M.J.D., *"Introduction to constrained Optimization,"* In: 'Numerical mehtods for Constrained Optimization,' P.E. Gill and W. Murray, (eds). Academic Press, London and Newyork, pp. 1-28, 1974.

[16] Powell, M.J.D., *"Variable Metric Methods for Constrained Optimization,"* In: 'Mathematical Programming: The state of art,' A. Bachem, M. Grotschel and B. Korte,(eds), Springer-Verlag, Berlin , Heidelberg, Newyork and Tokyo, pp. 288-311, 1983.

[17] Rao, S.S., *"Optimization, Theory and Application,"* Wiley-Eastern limited, NewDelhi, 1984

[18] Subramenya, K., *"Flow In Open Channels,"* Tata McGraw-Hill Publishing Company Limited, New Delhi, 1984.

[19] Urquhart, W.J., "Hydraulics," Engineerng Field Manual, *U.S. Department of Agriculture, Soil Conservation Service* Washington, 1975.

[20] WasanthaLal, A.M., "Calibration of Riverbed Roughness," *Journal of Hydraulic Engineering,* Vol. 121, No. 9, Sep., 1995, pp. 664-671.

[21] Yeh, W.W-G., and Tauxe, G,W., "Optimal Identification of Aquifer Diffusivity Using Quasilinearisation," *Water Resources Research,* Vol. 7, No. 4, Aug., 1971, pp. 955-962.